2.3. Large Scale Channel Modeling — Shadowing

Reading assignment

- 4.9.

1 Multipath Channel

Because of the mobility and complex environment, there are two types of channel variations: 1) small scale variation which is due to primarily the random phase delay of multiple paths; 2) large scale variation due to the distance between Tx and Rx as well as the physical environment of a particular cell. This separation is illustrated in Figure 1 as we have seen before.

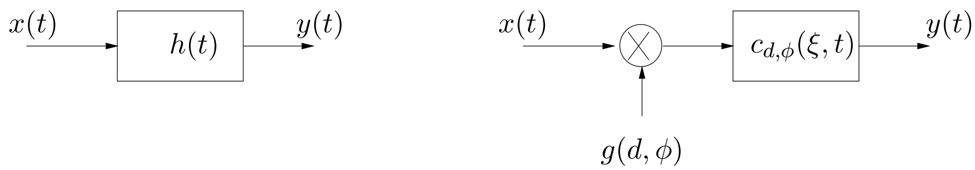

Figure 1: Small-scale and large-scale variation components of mobile channel.

In Figure 1 $c_{d,\phi}(\xi,t)$ is normalized with average power gain equal to 1, i.e.,

$$c(\xi,t)=\sum_n \beta_n(d,\phi)e^{j\phi_n(d,\phi)}\delta(t-\tau_n(d,\phi))$$

where $\sum_n \beta_n^2(d,\phi)=1$. The large-scale variations in average amplitude gain are then lumped into the multiplicative term $g(d,\phi)$. Thus the corresponding power gain for the large-scale variations is

$$G(d,\phi)=g^2(d,\phi)$$

We further decouple $G(d,\phi)$ into two parts

$$G(d,\phi)=S(d,\phi)\overline{G(d)}$$

where $\overline{G(d)}$ is a deterministic function of d that accounts for the power-law decaying of signals; while $S(d,\phi)$ accounts for the large-scale time variation, i.e., shadowing. Equivalently we can write in dB as

$$G(d,\phi)[dB]=Z(d,\phi)[dB]+K(d)[dB] \tag{1}$$

where $Z(d,\phi)=10\log S(d,\phi)$ and $K(d)=10\log\overline{G(d)}$. We will now study the model for $S(d,\phi)$ (or $Z(d,\phi)[dB]$), i.e., the shadow fading model.

2 Shadowing

We have studied $K(d)$, the power-law path loss function and developed methods for parameter estimation associated with the model. We emphasize that $K(d)$ accounts for the (deterministic) average large-scale power gain. The variation of this gain is reflected in the term $Z(d, \phi)$. This variation is called the shadow fading, or shadowing. The name is derived from the phenomenon that a MS experiences (slow) power gain variation when it travels in and out of the shadow of large buildings. We will give detailed accounts on the first order shadowing model — lognormal shadowing. We also briefly mention about second order statistics for shadowing that take into account the spatial correlation among various locations.

2.1 First order statistics for shadow fading — log-normal shadowing

The most commonly used model for $Z(d, \phi)$ is a zero mean Gaussian random variable with variance σ_Z^2. The zero mean is an obvious choice — any non zero should be lumped into the path loss model $K(d)$. As to the Gaussian assumption, it has its root on the central limit theorem which states, loosely, that the summation of a large number of independent random variables will approach a Gaussian random variable. Notice that $Z = 10 \log S$, and under the Gaussian assumption on Z, it can be easily derived that the pdf for S is

$$p_S(s) = \frac{10}{s \ln(10)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(10 \log s)^2}{2\sigma_Z^2}\right) u(s)$$

where $u(s)$ is the unit step function. This is the pdf for a lognormal random variable (its base-10 logarithm follows normal distribution). Notice that the extra factor as compared with the lognormal in HW#2 Problem 2 is due to the difference between natural logarithm and base-10 logarithm.

We have obtained the mean value for lognormal distribution which turns out to be different than $\ln \mu$ (or $\log \mu$ for base-10 logarithm) where μ is the mean of the corresponding normal distribution. It is interesting though to note that $\ln \mu$ (or $\log \mu$) is indeed the median of lognormal distribution. This is why $K(d)$ is sometime called median link gain (While you may know the answer to Problem 2(b), you still need to DERIVE it in your homework hand-in).

Lognormal shadowing describes the marginal distribution of the shadow fading statistics. Because of the Gaussian assumption, it allows easy calculation of some of the important quantities in link budget design. Below we investigate three important and related applications

1. Outage probability

   The first application is the computation of the so-called outage probability, defined as the probability that the received signal power falls below a certain acceptable threshold, say γ. Given that Z is $\mathcal{N}(0, \sigma_Z^2)$, then $G(d, \phi)[dB] = Z(d, \phi) + K(d)$ is then Gaussian distributed with mean $K(d)$ and variance σ_Z^2. We define

$$Q(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-t^2/2} dt$$

   as the complementary distribution function of standard Gaussian ($\mathcal{N}(0, 1)$). Then we have

$$\begin{aligned} P[outage] &= P[G(d, \phi)[dB] < \gamma] \\ &= P\left(\frac{G(d, \phi)[dB] - K(d)}{\sigma_Z} < \frac{\gamma - K(d)}{\sigma_Z}\right) \\ &= 1 - Q\left(\frac{\gamma - K(d)}{\sigma_Z}\right) \end{aligned}$$

From the previous lecture, we know that

$$K(d) = A - 10n \log d$$

Therefore

$$P[outage] = 1 - Q\left(\frac{\gamma - A + 10n \log d}{\sigma_Z}\right)$$

Calculating $Q(\cdot)$ using Matlab

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

Change of variable $s = \sqrt{2}t$, hence $t = s/\sqrt{2}$. Therefore equivalently

$$\begin{aligned} erfc(x) &= \frac{2}{\sqrt{\pi}} \int_{\sqrt{2}x}^{\infty} e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2}} ds \\ &= 2 \int_{\sqrt{2}x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \\ &= 2Q(\sqrt{2}x) \end{aligned}$$

Therefore

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right)$$

2. Cellular Area Reliability Function

Another application of lognormal shadowing is to allow the calculation of the so-called cellular area reliability function.

Definition 1 Area Reliability Function *Given a particular service area, e.g., a cell, the percentage of area with a received signal that is greater than a certain threshold γ is called area reliability function and is denoted as $U(\gamma)$.*

For an area with radius R, reliability function and the outage probability is related by

$$\begin{aligned} U(\gamma) &= 1 - \frac{1}{\pi R^2} \int P_{outage}(\rho, \phi) dA \\ &= 1 - \frac{1}{\pi R^2} \int_0^R \int_{-\pi}^{\pi} P_{outage}(\rho, \phi) \rho d\rho d\phi \\ &= \frac{1}{\pi R^2} \int_0^R \int_{-\pi}^{\pi} P[G(d, \phi)[db] > \gamma] \rho d\rho d\phi \\ &= \frac{1}{\pi R^2} \int_0^R \int_{-\pi}^{\pi} Q\left(\frac{\gamma - A + 10n \log d}{\sigma_Z}\right) \rho d\rho d\phi \\ &= \frac{2}{R^2} \int_0^R Q(a + b \ln \rho) \rho d\rho \end{aligned}$$

where

$$a = \frac{\gamma - A}{\sigma_Z} \qquad b = \frac{10n}{\sigma_Z^2 \ln 10}$$

To continue,

$$\begin{aligned}
U(\gamma) &= \frac{2}{R^2}\int_0^R Q(a+b\ln\rho)\rho d\rho \\
&= \frac{1}{R^2}\int_0^R Q(a+b\ln\rho)d\rho^2 \\
&= \frac{1}{R^2}Q(a+b\ln R)\rho^2|_0^R - \frac{1}{R^2}\int_0^R \rho^2 d(Q(a+b\ln\rho)) \\
&= Q(a+b\ln R) - \frac{1}{R^2}\int_0^R \rho^2 d(Q(a+b\ln\rho))
\end{aligned}$$

Given that

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

we get that

$$d(Q(a+b\ln\rho) = -\frac{1}{\sqrt{2\pi}}e^{-\frac{(a+b\ln\rho)^2}{2}}\frac{d}{d\rho}(a+b\ln\rho)d\rho = -\frac{b}{\sqrt{2\pi}\rho}e^{-\frac{(a+b\ln\rho)^2}{2}}d\rho$$

Therefore

$$\begin{aligned}
U(\gamma) &= Q(a+b\ln R) + \frac{1}{R^2}\int_0^R \frac{b\rho}{\sqrt{2\pi}}e^{-\frac{(a+b\ln\rho)^2}{2}}d\rho \\
&\overset{y=a+b\ln\rho}{=} Q(a+b\ln R) + \frac{1}{R^2}\int_{-\infty}^{a+b\ln R} e^{\frac{2}{b^2}-\frac{2a}{b}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-2/b)^2}{2}\right)dy \\
&= Q(a+b\ln R) + \frac{e^{\frac{2}{b^2}-\frac{2a}{b}}}{R^2}\left[1-Q\left(a+b\ln R-\frac{2}{b}\right)\right]
\end{aligned}$$

which provides a closed-form solution to $U(\gamma)$.

3. Area Reliability versus Edge Reliability

   Given the lognormal shadowing and path loss model, it is easy to compute the outage probability at the edge of the cell. The reliability function at the edge of a cell with radius R is indeed a simple $Q(\cdot)$ function.

   $$E(\gamma) = 1 - P_{outage}(R) = Q(a+b\ln R)$$

   where a and b are as defined before.

   It is often desirable to study the relationship between $E(\gamma)$ and $U(\gamma)$. It turns out that given $E(\gamma)$, $U(\gamma)$ is only a function of b, i.e., the ratio between the path loss exponent and the variance for shadow fading. That is, given $E(\gamma)$, $U(\gamma)$ is independent to both A and R. Proof is left as an exercise (HW#2 Problem 4).

Other applications include statistical analysis of signal to interference ratio. Notice that the previous co-channel interference analysis is based on deterministic path loss model. More realistic model should also incorporate the shadow fading in the received signal power both from the desired transmitter and interferers. Thus the SIR is a ratio between two random variables instead of two constants that are only functions of geographical distances. See chapter 3 in Stüber [2] for a detailed analysis.

2.2 Second order statistics for shadow fading

The first order statistics, i.e., the marginal distribution for shadow fading is studied and normal distribution is used to model $Z(d, \phi)$ that results in lognormal shadowing. Second order statistics study the joint distribution between different points within a cell. If joint Gaussian is assumed for $Z(d_1, \phi_1)$ and $Z(d_2, \phi_2)$ (notice that marginal Gaussian does not imply joint Gaussian), then we need only specify the correlation function as it uniquely determines the joint probability distribution function for joint Gaussian random variables. Notice that the mean values are always zero.

To study the correlation between different points of a service area, it is more convenient to use Cartesian coordinates (x, y) instead of (d, ϕ). The received signal, considered as a random process as a function of two dimensional index (x and y), is called a random field. In particular, we simplify our model by assuming a homogeneous and isotropic random field, i.e., the joint distribution is invariant to any rigid body motions. Specifically we have the following two definitions.

Definition 2 *A random field $V(x, y)$ is said to be homogeneous (or stationary) if for all n, and all choices of $(x_1, y_2), \cdots, (x_n, y_n)$ and (s_x, s_y), the joint distribution of $V(x_1, y_1), \cdots, V(x_n, y_n)$ is the same as that of $V(x_1 + s_x, y_2 + s_y), \cdots, V(x_n + s_x, y_n + s_y)$.*

A homogeneous random field is translation invariant. Therefore the joint distribution between the fields at two points, say (x_1, y_1) and (x_2, y_2) is a function only of the difference vector $(x_1 - x_2, y_1 - y_2)$.

Definition 3 *A random field $V(x, y)$ is said to be homogeneous and isotropic if for all n, and all choices of $(x_1, y_2), \cdots, (x_n, y_n)$ and (s_x, s_y), the joint distribution of $V(x_1, y_1), \cdots, V(x_n, y_n)$ is the same as that of $V(\mathcal{T}_1(x_1, y_1), \cdots, V(\mathcal{T}_n(x_n, y_n)$, where $\mathcal{T}_k$'s are any transformations on the points that preserve the distance between any pair of these points.*

Thus an homogeneous and isotropic field is invariant to any rigid body motion. A direct consequence is that the joint distribution between the fields at two points, say (x_1, y_1) and (x_2, y_2) is a function only of the distance between the two points, defined as

$$\Delta = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This is in analogous to the stationarity of a random process. Under Gaussian assumption, as is the case for $Z(x, y)$, this is equivalent to say that the correlation is completely characterized by Δ, i.e.,

$$R_Z((x_1, y_1), (x_2, y_2)) = R_Z(\Delta)$$

A naive model for correlation function is the white model, that is, the correlation is a delta function thus any pair of points are uncorrelated with each other. This is in general unrealistic since points that are close to each other are usually correlated to each other, therefore a more commonly used model, first proposed by Gudmundson [1], is the exponential correlation model

$$R_Z(\Delta) = \sigma_Z^2 \exp\left(-\frac{|\Delta|}{D_c}\right)$$

where D_c is defined to be the correlation distance of the fading process. Thus the correlation between two points decreases exponentially as the distance increases.

Example 1 *If a user is traveling through a cell, the received signal he sees is a random process instead of a random field, and given the specification of the random field, one should be able to find the statistics for the random process from the moving MS's perspective. See HW#3 Problem 5 for an example.*

Assume that a MS is moving across a cell along the solid line with a constant velocity $v = 10m/s$ *as shown in Figure 2. The cell radius* $R = 1km$ *and the parameters for the path loss model are* $n = 3$ *and* $A = 140dBm$*. The variance for shadow fading is* $\sigma_Z = 8dBm$*. We set the time that the MS enters the cell (point A) as* $t_A = 0$*. The received signal power for the MS is then a random process, denoted as* $P(t)$*.*

1. *Find* t_B*, the time when the MS reaches point B.*
2. *Find the average received power as a function of* t *for* $0 \leq t \leq t_B$*. Notice this is the mean value of the perceived random process from the user's perspective, i.e.,* $E[P(t)]$*.*
3. *Using the homogeneous and isotropic random field assumption with exponential correlation (with correlation distance* D_c*), determine the correlation between the* $P(t_1)$ *and* $P(t_2)$ *with* $0 < t_1 < t_2 < t_B$*.*
4. *What is the outage probability at point A? Point B? Halfway between A and B? The acceptable received power threshold is assumed to be* $20dBm$*.*

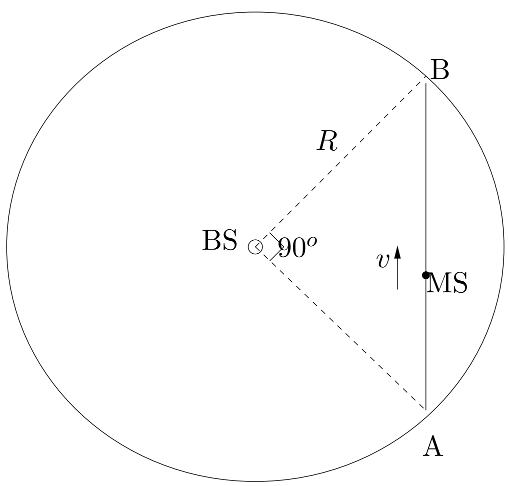

Figure 2: A user moves along the solid line.

References

[1] M. Gudmundson. Correlation model for shadow fading in mobile radio systems. *Electron. Letter*, 27:2145–2146, 1991.

[2] G.L. Stüber. *Principles of Mobile Communication*. Kluwer, Boston, MA, 2nd edition, 2001.