The Gaussian Multiple Access Wire-Tap Channel

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Abstract

We consider the Gaussian Multiple Access Wire-Tap Channel (GMAC-WT). In this scenario, multiple users communicate with an intended receiver in the presence of an intelligent and informed wire-tapper who receives a degraded version of the signal at the receiver. We define a suitable security measure for this multi-access environment. An outer bound for the rate region such that secrecy to some pre-determined degree can be maintained is derived. Using Gaussian codebooks, an achievable such secrecy region is also identified. Gaussian codewords are shown to achieve the sum capacity outer bound, and the achievable region coincides with the outer bound for Gaussian codewords, giving the capacity region when inputs are constrained to be Gaussian. Numerical results showing the new rate region are presented and compared with the capacity region of the Gaussian Multiple-Access Channel (GMAC) with no secrecy constraints. It is shown that the multiple-access nature of the channel can be utilized to reduce the rate lost due to the secrecy constraint for each user.

Index Terms

Secrecy Capacity, Gaussian Multiple Access Channel, Wire-Tap

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I. INTRODUCTION

Shannon, in [1], analyzed secrecy systems in communications and he showed that to achieve perfect secrecy of communications, we must have the conditional probability of the *cryptogram* given a message independent of the actual transmitted message.

In [2], Wyner applied this concept to the discrete memoryless channel, with a wire-tapper who has access to a degraded version of the intended receiver's signal. He measured the amount of "secrecy" using the conditional entropy Δ , the conditional entropy of the transmitted message given the received signal at the wire-tapper. The region of all possible (R, Δ) pairs is determined, and the existence of a *secrecy capacity*, C_s , for communication below which it is possible to transmit zero information to the wire-tapper is shown [2].

Carleial and Hellman, in [3], showed that it is possible to send several low-rate messages, each completely protected from the wire-tapper individually, and use the channel at close to capacity. The drawback is, in this case, if any of the messages are revealed to the wire-tapper, the others might also be compromised. In [4], the authors extended Wyner's results to Gaussian channels and also showed that Carleial and Hellman's results in [3] also held for the Gaussian channel [4]. Csiszár and Körner, in [5], showed that Wyner's results can be extended to weaker, so called "less noisy" and "more capable" channels. Furthermore, they analyzed the more general case of sending common information to both the receiver and the wire-tapper, and private information to the receiver only.

More recently, the closely related problem of common randomness and secret key generation has gathered attention. Maurer, [6], and Bennett et. al., [7], have focused on the process of "distilling" a secret key between two parties in the presence of a wire-tapper. In this scenario, the wire-tapper has partial information about a common random variable shared by the two parties, and the parties use their knowledge of the wire-tapper's limitations to slowly distill a secret key. [7] breaks this down into three main steps: (i) advantage distillation: where the two parties have zero wiretap capacity and need to find some way of creating an advantage over the wire-tapper, (ii) information reconciliation: where the secret key decided by one of the partners is communicated to the other partner and the wire-tapper is still left with only partial information about it, (iii) privacy amplification: where a new secret key is generated from the previous one about which the wire-tapper has negligible information. In [6], it was shown that for the case when the wire-tap channel capacity is zero between two users, the existence of a "public" feedback channel that the wire-tapper can also observe can nevertheless enable the two parties to be able to generate a secret key with perfect secrecy. This discussion was then furthered by [8] and [9] where the secrecy key capacities and *common randomness* capacities, the maximum rates of common randomness that can be generated by two terminals, were developed for several models. It was also argued in [10], that the secrecy constraint developed by Wyner and later utilized by Csiszár and Körner was "weak" since it only constrained the rate of information leaked to the wire-tapper, rather than the total information. It was shown that Wyner's scenario could be extended to "strong" secrecy with no loss in achievable rates, where the secrecy constraint is placed on the total information obtained by the wire-tapper, as the information of interest might be in the small amount leaked. This corresponds to making the leaked information go to zero exponentially rather than just inversely with n. Maurer then examined the case of active adversaries, where the wire-tapper has read/write access to the channel in [11]-[13]. Venkatesan and Anantharam examined the cases where the two terminals generating common randomness were connected by different DMC's in [14] and later generalized this to a network of DMC's connecting any finite number of terminals in [15]. Csiszár and Narayan extended Ahslwede and Csiszár's previous work to multiple-terminals by looking at what a helper terminal can contribute in [16] and the case of multiple terminals where an arbitrary number of terminals are trying to distill a secret key and a subset of these terminals can act as helper terminals to the rest in [17].

In this paper, we consider the Gaussian Multiple Access Channel (GMAC) and define two separate secrecy constraints, which we call the *individual* and *collective* secrecy constraints. These two different sets of security constraints are (i) the normalized entropy of any set of messages conditioned on the transmitted codewords of the other users and the received signal at the wire-tapper, and (ii) the normalized entropy of any set of messages conditioned on the wire-tapper's received signal. The first set of constraints is more conservative to ensure secrecy of any subset of users even when the remaining users are compromised. The second set of constraints ensures the collective secrecy of any set of users, utilizing the secrecy of the remaining users. In [18], we concerned ourselves mainly with the *perfect secrecy rate region* for both sets of constraints. In this paper, we consider the general case where a pre-determined level of secrecy is provided. Under these constraints, we find outer bounds for the secure rate region. Using random Gaussian codebooks, we find achievable secure rate regions for each constraint, where users can communicate with arbitrarily small probability of error with the intended receiver, while the wire-tapper is kept ignorant to a pre-determined level. We show that using the "collective secrecy constraints", when we limit ourselves to using Gaussian codebooks, these bounds coincide and give the capacity region for Gaussian codebooks. Furthermore, it is shown that Gaussian codebooks achieve sum capacity for the GMAC-WT using simultaneous superposition coding, [19]. We also show that a simple TDMA scheme using the results of [4] for the single-user case also achieves sum capacity, but provides a strictly smaller region than shown in this paper. We also find an achievable region for the set of "individual constraints" which is obtained as a union of this TDMA scheme with superposition encoding. This region is smaller than the outer bounds, but achieves sum capacity and is close when the eavesdropper's channel is much noisier than the legitimate receiver's.

The rest of the paper is organized as follows: In section II, we present a summary of our main results. Section III describes the system model and the problem statement. Sections IV and V describe the outer bounds and the achievable rates, respectively. Section VI gives our numerical results followed by our conclusions and future work. All necessary proofs are given in the Appendices.

II. MAIN RESULTS

In this paper, we

- 1) Define two sets of information theoretic secrecy measures for a multiple-access channel:
 - Individual: Secrecy must be maintained for any user even if the remaining users are compromised.
 - Collective: Secrecy is achieved with the assumption that all users are secure.
- Find outer bounds for both sets of constraints and show that the sum capacity bound is the same for both sets of constraints.
- 3) Using Gaussian codebooks, find achievable regions for both sets of constraints.
 - For individual constraints, the achievable region is a subset of the outer bounds, but using TDMA it is possible to achieve the sum capacity.
 - For collective constraints, if we are limited to Gaussian codebooks, then the inner and outer bounds coincide. In addition, it is shown that Gaussian codebooks achieve the sum capacity. The achievable region found is a superset of the TDMA region.
- 4) Show that the achievable region for the collective secrecy constraints is larger than the capacity region for the individual constraints. Thus, it is found that it is possible to use the multi-access nature of the channel to our advantage and enlarge the secrecy region than is possible when we want to achieve secrecy for each user separately.

III. SYSTEM MODEL AND PROBLEM STATEMENT

We consider K users communicating with a receiver in the presence of a wire-tapper. Transmitter j chooses a message W_j from a set of equally likely messages $\mathcal{W}_j = \{1, \ldots, M_j\}$. The messages are encoded using $(2^{nR_j}, n)$ codes into $\{\tilde{X}_j^n(W_j)\}$, where $R_j = \frac{1}{n} \log_2 M_j$. The encoded messages $\{\tilde{\mathbf{X}}_j\} = \{\tilde{X}_j^n\}$ are then transmitted, and the intended receiver and the wiretapper each get a copy $\mathbf{Y} = Y^n$ and $\mathbf{Z} = Z^n$. The receiver decodes \mathbf{Y} to get an estimate of the transmitted messages, $\hat{\mathbf{W}}$. We would like to communicate with the receiver with arbitrarily low probability of error, while maintaining perfect secrecy, the exact definition of which will be made precise shortly.

The signals at the intended receiver and the wiretapper are given by

$$\mathbf{Y} = \sum_{k=1}^{K} \sqrt{h_k^{(\mathsf{M})}} \tilde{\mathbf{X}}_k + \tilde{\mathbf{N}}^{(\mathsf{M})}$$
(1)

$$\mathbf{Z} = \sum_{k=1}^{K} \sqrt{h_k^{(\mathsf{W})}} \tilde{\mathbf{X}}_k + \tilde{\mathbf{N}}^{(\mathsf{W})}$$
(2)

where $\tilde{\mathbf{N}}^{(M)}, \tilde{\mathbf{N}}^{(W)}$ are the AWGN. Each component of $\tilde{\mathbf{N}}^{(M)} \sim \mathcal{N}(0, \sigma_{\mathsf{M}}^2)$ and $\tilde{\mathbf{N}}^{(W)} \sim \mathcal{N}(0, \sigma_{\mathsf{W}}^2)$. We also assume the following transmit power constraints:

$$\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_{ji}^{2} \le \tilde{P}_{j,max}, \ j = 1, \dots, K$$
(3)

Similar to the scaling transformation to put an interference channel in standard form, [20], we can represent any GMAC-WT by an equivalent standard form:

$$\mathbf{Y} = \sum_{k=1}^{K} \mathbf{X}_k + \mathbf{N}^{(\mathsf{M})}$$
(4a)

$$\mathbf{Z} = \sum_{k=1}^{K} \sqrt{h_k} \mathbf{X}_k + \mathbf{N}^{(\mathsf{W})}$$
(4b)

where

- the original codewords $\{\tilde{\mathbf{X}}\}$ are scaled to get $\mathbf{X}_k = \sqrt{\frac{h_k^{(M)}}{\sigma_M^2}} \tilde{\mathbf{X}}_k$.
- The wiretapper's new channel gains are given by $h_k = \frac{h_k^{(W)} \sigma_M^2}{h_k^{(M)} \sigma_W^2}$.
- The noises are normalized by $\mathbf{N}^{(M)} = \frac{1}{\sigma_M^2} \tilde{\mathbf{N}}^{(M)}$ and $\mathbf{N}^{(W)} = \frac{1}{\sigma_W^2} \tilde{\mathbf{N}}^{(W)}$.

In this paper, we will be examining the special case of the wire-tapper getting a degraded version of the received signal. It can easily be shown that the wire-tapper gets a degraded version of the receiver's signal if and only if $h_1 = \ldots = h_K \equiv h < 1$. This is equivalent to the wire-tapper's received signal being a noisier version of the legitimate receiver's scaled received signal,

$$\mathbf{Z} = \sqrt{h}\mathbf{Y} + \mathbf{N}^{(\mathsf{W}-\mathsf{M})} \tag{5}$$

where $\mathbf{N}^{(W-M)}$ has each component $\sim \mathcal{N}(0, 1-h)$. This model is illustrated in Figure 1. In practical situations, we can think of this as the wire-tapper being outside of a controlled indoor environment, such as in [21] or just being able to wire-tap the receiver rather than receive the signals itself.

A. The Secrecy Measures

We aim to provide each user with a pre-determined amount of secrecy. Letting Δ_S be our secrecy constraint for any subset S of users, we require that $\Delta_S \ge \delta$ for all sets $S \subseteq K$, with $\delta \in [0, 1]$ as the required level of secrecy. $\delta = 1$ corresponds to *perfect secrecy*, where the wiretapper is not allowed to get any information; and $\delta = 0$ corresponds to no secrecy constraint. To that end, we use an approach similar to [4], and define two sets of secrecy constraints using the normalized equivocations for sets of users. These are:

1) Individual Secrecy: Let us first define

$$\Delta_k^{(I)} \triangleq \frac{H(W_k | \mathbf{X}_{k^c}, \mathbf{Z})}{H(W_k)} \quad \forall k = 1, ..., K$$
(6)

where k^c is the set of all users except user k. $\Delta_k^{(I)}$ denotes the normalized entropy of a user's message given the received signal at the wire-tapper as well as all other users' transmitted symbols. As our secrecy criterion, we require that each user $k = 1, \ldots, K$ satisfy $\Delta_k^{(I)} \ge \delta$. This constraint guarantees that information obtained at the wire-tapper about the user k's signal is limited even if all other users are compromised. Let us define the secrecy measure for a subset of users, $S \subseteq \mathcal{K} = \{1, \ldots, K\}$, as

$$\Delta_{\mathcal{S}}^{(I)} \triangleq \frac{H(\mathbf{W}_{\mathcal{S}} | \mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z})}{H(\mathbf{W}_{\mathcal{S}})} \quad \forall \mathcal{S} \subseteq \mathcal{K} = \{1, \dots, K\}$$
(7)

where $\mathbf{W}_{\mathcal{S}} = \{W_j\}_{j \in \mathcal{S}}.$

If the individual secrecy constraints for all users in the set S are satisfied, then the constraint for set S is also satisfied. To see this, without loss of generality, let S = 1, ..., S where $S \leq K$. We can write

$$H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}},\mathbf{Z}) = \sum_{j=1}^{S} H(W_{j}|W^{j-1}\mathbf{X}_{\mathcal{S}^{c}}\mathbf{Z})$$
(8)

$$\geq \sum_{j=1}^{S} H(W_j | W^{j-1} \mathbf{X}_{j^c}, \mathbf{Z})$$
(9)

$$=\sum_{j=1}^{S} H(W_j | \mathbf{X}_{j^c}, \mathbf{Z})$$
(10)

$$\geq \sum_{j=1}^{S} \delta H(W_j) \tag{11}$$

$$=\delta H(\mathbf{W}_{\mathcal{S}}) \tag{12}$$

where (9) follows using conditioning, (10) is due to the fact that W_j is conditionally independent of all W_k given \mathbf{X}_k, \mathbf{Z} . (11) comes from our assumption that for all $j \in \mathcal{S}, \Delta_j^{(I)} \geq \delta$. Thus, for any set of users $\mathcal{S} \subseteq \mathcal{K} = \{1, \ldots, K\}$, the individual secrecy constraints $\{\Delta_j^{(I)} \geq \delta\}$ for all users j in the subset \mathcal{S} also guarantees the joint perfect secrecy of the set \mathcal{S} , i.e., $\Delta_{\mathcal{S}}^{(I)} \geq \delta$.

2) *Collective Secrecy:* Clearly (7) is a conservative measure, following closely from the singleuser measure adopted in [4]. Let us now define a revised secrecy measure to take into account the multi-access nature of the channel.

$$\Delta_{\mathcal{S}}^{(C)} \triangleq \frac{H(\mathbf{W}_{\mathcal{S}} | \mathbf{Z})}{H(\mathbf{W}_{\mathcal{S}})} \quad \forall \mathcal{S} \subseteq \mathcal{K}$$
(13)

Using this constraint guarantees that each subset of users maintains a level of secrecy greater than δ . Since this must be true for all sets of users, collectively the system has at least the same level of secrecy. However, if a group of users are somehow compromised, the remaining users may also be vulnerable. We require the secrecy constraint to be satisfied separately for each $S \subseteq \mathcal{K}$, since otherwise it is possible to have $\Delta_S^{(C)} \ge \delta$, but $\Delta_{\mathcal{J}}^{(C)} < \delta$ for some $\mathcal{J} \subset S$.

B. The δ -secret rate region

Definition 1 (Achievable rates with δ -secrecy). The rate *K*-tuple $\mathbf{R} = (R_1, \ldots, R_K)$ is said to be *achievable with* δ -secrecy if for any given $\epsilon > 0$ there exists a code of sufficient length *n* such that

$$\frac{1}{n}\log_2 M_k \ge R_k - \epsilon \quad k = 1, \dots, K$$
(14)

$$P_e \le \epsilon \tag{15}$$

$$\Delta_{\mathcal{S}} \ge \delta \quad \forall \mathcal{S} \subseteq \mathcal{K} \tag{16}$$

where user k chooses one of M_k symbols to transmit according to the uniform distribution, $\Delta \in \{\Delta^{(I)}, \Delta^{(C)}\}$ and

$$P_e = \frac{1}{\prod_{k=1}^{K} M_k} \sum_{\mathbf{w} \in \times_{k=1}^{K} W_k} \Pr\{\hat{\mathbf{W}} \neq \mathbf{W} | \mathbf{W} \text{ was sent}\}.$$
 (17)

is the average probability of error. We will call the set of all achievable rates with δ -secrecy, the δ -secret rate region, and denote it C_{δ} , where $C_{\delta} \in \{C_{\delta}^{(I)}, C_{\delta}^{(C)}\}$.

C. Some Preliminary Definitions

Before we state our results, we define the following quantities for any $S \subseteq \mathcal{K}$.

$$P_{\mathcal{S}} \triangleq \sum_{k \in \mathcal{S}} P_{k} \qquad \qquad R_{\mathcal{S}} \triangleq \sum_{k \in \mathcal{S}} R_{k}$$
$$C_{\mathcal{S}}^{(\mathsf{M})} \triangleq C\left(P_{\mathcal{S}}\right) \qquad \qquad \tilde{C}_{\mathcal{S}}^{(\mathsf{W})} \triangleq C\left(hP_{\mathcal{S}}\right) \qquad \qquad \tilde{C}_{\mathcal{S}}^{(\mathsf{W})} \triangleq C\left(\frac{hP_{\mathcal{S}}}{hP_{\mathcal{S}^{c}}+1}\right)$$

where $C(\xi) \triangleq \frac{1}{2}\log(1+\xi)$ and $S^c = \mathcal{K} \setminus S$. The quantities with $S = \mathcal{K}$ will sometimes also be used with the subscript *sum*.

IV. Outer Bounds on the δ -Secret Rate Region

In this section, we present outer bounds on the sets of achievable δ -secret rates, denoted C_{δ} , and explicitly state the outer bound on the achievable sum-rate with δ -secrecy. We also evaluate these bounds assuming we are limited to using Gaussian codebooks for calculation purposes, $\bar{\mathcal{G}}_{\delta}$. We see that $\bar{\mathcal{C}}_{\delta}^{(I)} \equiv \bar{\mathcal{G}}_{\delta}^{(I)}$, and show that $\bar{\mathcal{G}}_{\delta}^{(C)}$ achieves the maximum sum rate of $\bar{\mathcal{C}}_{\delta}^{(C)}$.

A. Individual Secrecy

Theorem 2. For the GMAC-WT, the secure rate-tuples (R_1, \ldots, R_K) such that $\Delta_{\mathcal{S}}^{(I)} \ge \delta, \forall \mathcal{S} \subseteq \mathcal{K}$ must satisfy

$$R_{\mathcal{S}}(\delta) \le \min\left\{C_{\mathcal{S}}^{(\mathsf{M})}, \quad \frac{1}{\delta}\left[C_{\mathcal{S}}^{(\mathsf{M})} - C_{\mathcal{S}}^{(\mathsf{W})}\right]\right\} \quad \forall \mathcal{S} \subseteq \mathcal{K}$$
(18)

The set of all such rate vectors will be denoted $\bar{C}^{(I)}_{\delta}$.

Corollary 2.1. The sum capacity with δ -secrecy satisfies

$$C_{sum}^{(I)}(\delta) \le \bar{C}_{sum}(\delta) \triangleq \min\left\{C_{sum}^{(\mathsf{M})}, \frac{1}{\delta}[C_{sum}^{(\mathsf{M})} - C_{sum}^{(\mathsf{W})}]\right\}$$
(19)
pendix I-A.

Proof: See Appendix I-A.

B. Collective Secrecy

Our main result is presented in the following theorem:

Theorem 3. For the GMAC-WT, the secure rate-tuples (R_1, \ldots, R_K) such that $\Delta_S \ge \delta$, $\forall S \subseteq K$ must satisfy

$$R_{\mathcal{S}}(\delta) \le \min\left\{C_{\mathcal{S}}^{(\mathsf{M})}, \quad \frac{1}{\delta} \left[C_{\mathcal{S}}^{(\mathsf{M})} - C\left(\frac{h\sum_{j\in\mathcal{S}} 2^{\frac{2}{n}H(\mathbf{X}_j)}}{2\pi e \left(hP_{\mathcal{S}^c} + 1\right)}\right)\right]\right\}$$
(20)

The set of all **R** satisfying (20) is denoted $\bar{\mathcal{C}}_{\delta}^{(C)}$.

Corollary 3.1. The sum capacity with δ -secrecy satisfies

$$C_{sum}^{(C)}(\delta) \le \bar{C}_{sum}(\delta) \tag{21}$$

where $\bar{C}_{sum}(\delta)$ is as defined in Corollary 2.1.

Corollary 3.2. The rate-tuples with δ -secrecy using Gaussian codebooks must satisfy

$$R_{\mathcal{S}}(\delta) \le \min\left\{C_{\mathcal{S}}^{(\mathsf{M})}, \quad \frac{1}{\delta} \left[C_{\mathcal{S}}^{(\mathsf{M})} - \tilde{C}_{\mathcal{S}}^{(\mathsf{W})}\right]\right\} \quad \forall \mathcal{S} \subseteq \mathcal{K}$$
(22)

The set of all such **R** is denoted $\bar{\mathcal{G}}_{\delta}^{(C)}$.

Proof: See Appendix I-B.

Remark: Since $C_{\mathcal{K}}^{(W)} = \tilde{C}_{\mathcal{K}}^{(W)}$, Corollary 3.2 indicates that Gaussian codebooks have the same upper bound on sum capacity as given by Corollary 3.1. This is also the sum capacity bound for individual secrecy constraints from Corollary 2.1, as $\Delta_{\mathcal{K}}^{(I)} \equiv \Delta_{\mathcal{K}}^{(C)}$.

$$C_{sum}(\delta) \le \bar{C}_{sum}(\delta) \triangleq \min\left\{ C\left(P_{\mathcal{K}}\right), \frac{1}{\delta} [C\left(P_{\mathcal{K}}\right) - C\left(hP_{\mathcal{K}}\right)] \right\}$$
(23)

$$= \min\left\{ C\left(P_{\mathcal{K}}\right), \frac{1}{\delta}C\left(\frac{(1-h)P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right) \right\}$$
(24)

$$= \begin{cases} C(P_{\mathcal{K}}), & \text{if } \delta \leq \frac{C\left(\frac{(1-h)P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right)}{C(P_{\mathcal{K}})} \\ \frac{1}{\delta}C\left(\frac{(1-h)P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right), & \text{if } \delta \geq \frac{C\left(\frac{(1-h)P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right)}{C(P_{\mathcal{K}})} \end{cases}$$
(25)

$$= \begin{cases} C(P_{\mathcal{K}}), & \text{if } \delta \leq 1 - \frac{C(hP_{\mathcal{K}})}{C(P_{\mathcal{K}})} \\ \frac{1}{\delta}C\left(\frac{(1-h)P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right), & \text{if } \delta \geq 1 - \frac{C(hP_{\mathcal{K}})}{C(P_{\mathcal{K}})} \end{cases}$$
(26)

For perfect secrecy ($\delta = 1$), the second case is guaranteed and the sum rates limit becomes

$$\bar{C}_{sum}(1) = C\left(\frac{(1-h)P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right) = \frac{1}{2}\log\left(\frac{1+P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right) = C(P_{\mathcal{K}}) - C(hP_{\mathcal{K}}) = C_{sum}^{(\mathsf{M})} - C_{sum}^{(\mathsf{M})}$$
(27)

which is equal to the differences of the sum capacities of the legitimate receiver's and wiretapper's sum capacities. This result is in direct agreement with [4].

V. Achievable δ -Secret Rate Regions

In this section, we find a set of achievable rates using Gaussian codebooks, which we call $\underline{\mathcal{G}}_{\delta}$, and show that Gaussian codebooks achieve the limit on sum capacity. For collective secrecy

constraints, this region coincides with our previous upper bound evaluated using Gaussian codebooks, $\bar{\mathcal{G}}^{(C)}_{\delta}$, giving the full characterization of the δ -secret rate region using Gaussian codebooks, $\mathcal{G}^{(C)}_{\delta}$. For individual constraints, we find an achievable region that is the convex hull of the union of a region achieved similar to the collective constraints and a region achieved using TDMA. This resulting region is usually a strict subset of the outer bounds, but does achieve sum capacity.

A. Individual Secrecy

In [4], it has been shown that Gaussian codebooks can be used to maintain secrecy for a single user wire-tap channel. Using a similar approach, we show that an achievable region for perfect secrecy using individual constraints is given by:

Theorem 4. The following region is achievable with perfect secrecy for the GMAC-WT using Gaussian codebooks.

$$\underline{\mathcal{G}}_{\delta}^{(I)} = \left\{ \mathbf{R} \colon R_{\mathcal{S}} \le \min \left\{ C_{\mathcal{S}}^{(\mathsf{M})}, \frac{1}{\delta} \left(C_{\mathcal{S}}^{(\mathsf{M})} - \sum_{j \in \mathcal{S}} C_{j}^{(\mathsf{W})} \right) \right\} \quad \forall \mathcal{S} \subseteq \mathcal{K} \right\}$$
ee Appendix II-A.

Proof: See Appendix II-A.

In this case, the maximum sum rate achievable is given by

$$R_{sum}^{(I)}(\delta) = \min\left\{C_{sum}^{(\mathsf{M})}, \frac{1}{\delta}\left[C_{sum}^{(\mathsf{M})} - \sum_{j=1}^{K} C_{j}^{(\mathsf{W})}\right]\right\}$$
(29)

Observe that there is a reduction of $\sum_{j=1}^{K} C_{j}^{(W)} \ge C_{sum}^{(W)}$ in the sum rate due to secrecy constraints. This scheme, using stochastic encoding and Gaussian codebooks, achieves a sum rate that is less than the outer bound defined in (27). Also observe that the transmission of all the users with their maximum power may not be optimal for this case. To maximize the sum rate, we pose the following power allocation problem:

$$\max_{\mathbf{P}} R_{sum}^{(I)}(1) \triangleq \hat{R}(\mathbf{P}) = C_{sum}^{(\mathsf{M})} - \sum_{j=1}^{K} C_{j}^{(\mathsf{W})} \quad \text{s.t.} P_{j} \le P_{j,max}$$
(30)

since we know the first term in the minimum of (29) is maximized with $P_k = P_{k,max} \ \forall k \in \mathcal{K}$ which is on the border of the constraint set above. Let \mathbf{P}^* be the power allocation maximizing (30). Then, the maximum achievable sum rate is given simply by

$$R_{sum}^{(I)}(\delta) = \min\left\{C_{sum}^{(\mathsf{M})}, \frac{1}{\delta}\hat{R}(\mathbf{P}^*)\right\}$$
(31)

The solution to this problem is given by the theorem below:

Theorem 5. The sum capacity maximizing power allocation is such that:

- Optimum power allocation dictates that any given user transmits either with all its power or does not transmit, i.e., P_j = 0 or P_j = P_{j,max} for j = 1,..., K.
- The optimum set of users who are transmitting with full power, \mathcal{T} , should satisfy

$$\begin{split} & \frac{\partial C_{sum}}{\partial P_j} > 0 \quad \forall j \in \mathcal{T} \Rightarrow \sum_{k \in \mathcal{T}} P_{k,max} \le P_j + \frac{1}{h} - 1, \quad \forall j \in \mathcal{T} \\ & \frac{\partial C_{sum}}{\partial P_j} < 0 \quad \forall j \notin \mathcal{T} \Rightarrow \sum_{k \in \mathcal{T}} P_{k,max} \ge \frac{1}{h} - 1 \\ & \text{Appendix III-A} \end{split}$$

Proof: See Appendix III-A.

B. Collective Secrecy

Theorem 6. We can transmit with δ -secrecy using Gaussian codebooks at rates satisfying (22). The region containing all **R** satisfying these equations is denoted $\underline{\mathcal{G}}_{\delta}$.

Corollary 6.1. We can transmit with perfect secrecy ($\delta = 1$) using Gaussian codebooks at rates satisfying

$$R_{\mathcal{S}} \le C_{\mathcal{S}}^{(\mathsf{M})} - \tilde{C}_{\mathcal{S}}^{(\mathsf{W})} \tag{32}$$

Proof: See Appendix II-B.

C. Time-Division

We can also use a TDMA scheme to get an achievable region. Since, in such a scheme, only one user is transmitting at a given time, both sets of constraints collapse down to a set of

single-user secrecy constraints, for which the results were given in [4]:

Theorem 7. Consider this scheme: Let $\alpha_k \in [0, 1]$, k = 1, ..., K and $\sum_{k=1}^{K} \alpha_k = 1$. User k only transmits α_k of the time with power $P_{k,max}/\alpha_k$ using the scheme described in [4]. Then, the following set of rates is achievable:

$$\bigcup_{\substack{\mathbf{0} \preceq \boldsymbol{\alpha} \preceq \mathbf{1} \\ \sum_{k=1}^{K} \alpha_{k}=1}} \left\{ \mathbf{R} \colon R_{k} \leq \min\left\{\frac{\alpha_{k}}{\delta}C\left(\frac{(1-h)P_{k,max}}{\alpha_{k}+hP_{k,max}}\right), \alpha_{k}C\left(\frac{P_{k,max}}{\alpha_{k}}\right)\right\}, \quad k = 1, \dots, K \right\}$$
(33)

We will call the set of all **R** satisfying the above, $\underline{C}_{\delta}^{(T)}$.

Proof: Follows directly from [4, Theorem 1] and by noting that $C\left(\frac{P_{k,max}}{\alpha_k}\right) - C\left(\frac{hP_{k,max}}{\alpha_k}\right) = C\left(\frac{(1-h)P_{k,max}}{\alpha_k+hP_{k,max}}\right)$

Note that with this scheme, the sum capacity is given by

$$C_{sum}^{(T)}(\delta, \boldsymbol{\alpha}) = \sum_{k=1}^{K} \min\left\{\frac{\alpha_k}{\delta} C\left(\frac{(1-h)P_{k,max}}{\alpha_k + hP_{k,max}}\right), \alpha_k C\left(\frac{P_{k,max}}{\alpha_k}\right)\right\}$$
(34)

Theorem 8. The above described TDMA scheme achieves the upper limit on sum capacity, $\bar{C}_{sum}(\delta)$ using the optimum time-sharing parameters $\alpha_k = \frac{P_{k,max}}{\sum_{j=1}^{K} P_{j,max}}$.

Proof: See Appendix III-B.

Since in this scheme only one user is transmitting at any given time, both individual and collective constraints are satisfied. We will show that for collective secrecy constraints, this region is a subset of $\mathcal{Q}_{\delta}^{(C)}$. For individual secrecy constraints, however, this region is sometimes a superset of $\mathcal{Q}_{\delta}^{(I)}$, and sometimes a subset of $\mathcal{Q}_{\delta}^{(C)}$, but most of the time it helps enlarge this region. We can then, using time-sharing arguments, find a new achievable region for individual constraints that is the convex-closure of the union of the two regions, i.e.,

Proposition 9. The following region is achievable for individual secrecy constraints:

$$\mathcal{\underline{G}}_{\delta}^{(I_{\cup})} = \text{convex closure of } \left(\mathcal{\underline{G}}_{\delta}^{(I)} \cup \mathcal{\underline{G}}_{\delta}^{(T)} \right)$$
(35)

VI. NUMERICAL RESULTS

From (27), it can be seen that if the wire-tapper's degradedness is high (i.e., $h \to 0$), then $C_{sum}(1) \to C(P_{\mathcal{K}})$, we incur no loss in sum capacity and can still communicate with perfect secrecy as the sum capacity is achievable for both sets of constraints. On the other hand, if the wire-tapper's degradedness is low (i.e., $h \to 1$), then $C_{sum}(1) \to 0$ - it is no longer possible to communicate with perfect secrecy.

Another point of note is that as $C_{sum}(\delta) \leq \frac{1}{2\delta} \log \left(\frac{1+P_{\mathcal{K}}}{1+hP_{\mathcal{K}}}\right)$ and this term is an increasing function of $P_{\mathcal{K}}$, as $P_{\mathcal{K}} \to \infty$, $C_{sum}(\delta)$ is upper bounded by $-\frac{1}{2\delta} \log h$. We see that regardless of how much power we have available, the sum capacity with a non-zero level of secrecy is limited by the channel's degradedness, h, and the level of secrecy required, δ . Also, the sum capacity is inversely proportional to the level of secrecy desired, δ , but inversely proportional to the logarithm of h, the degradedness of the channel. Since in the range [0, 1], $\log(x)$ goes to 0 faster than $-x^{-1}$, an increase in h affects sum capacity more than a similar increase in δ . This can be seen by comparing Figures 13, 14 with Figures 17 and 18.

Figures 2–10 show the shapes of \mathcal{G}_{δ} for $\delta = 0.01, 0.5, 1$ and h = 0.1, 0.5, 0.9 for two users. When $\delta \rightarrow 0$, we are not concerned with secrecy, and the resulting region corresponds to the standard GMAC region, [22], see Figures 2–4. The region for $\delta = 1$ corresponds to the *perfect* secrecy region - transmitting at rates within this region, it is possible to send zero information to the wire-tapper, see Figures 8–10. The intermediate region, $\delta = 0.5$, can be thought of as constraining at least half the transmitted information to be secret. It can be seen that this enlarges the region from the perfect secrecy case, see Figures 5–7. It is shown that relaxing the secrecy constraint may provide a larger region, the limit of which is the GMAC region. Note that it is possible to send at capacity of the GMAC and still provide a non-zero level of secrecy, the minimum value of which depends on the level of degradedness, *h*. Also shown in the figures is the regions achievable by the TDMA scheme described in the previous section. Although TDMA achieves the sum capacity with optimum time-sharing parameters, this region is in general contained within \mathcal{G}_{δ} . Depending on h and δ , the TDMA region is sometimes a superset of $\mathcal{G}_{\delta}^{(I)}$ (Figures 2, 3, 5, 6, 8, 9), sometimes a subset of $\mathcal{G}_{\delta}^{(I)}$ (Figures 4,7), and sometimes the two regions can be used with time-sharing to enlarge the achievable region with individual constraints, see Figure 10. Close examination of Figures 12 and 18 shows that when the eavesdropper has a much "worse" channel, i.e., low h, and the secrecy constraint δ is low, then $\mathcal{G}_{\delta}^{(I)}$ gives a larger region. However, as we increase the secrecy constraint and the eavesdropper has a less noisy version of the intended receiver's signal, the TDMA region becomes more dominant.

Another interesting note is that even when a user does not have any information to send, it can still generate and send random codewords to confuse the eavesdropper and help other users. This can be seen in Figures 5, 6, 8–10 as the TDMA region does not end at the "legs" of \mathcal{G}_{δ} when \mathcal{G}_{δ} is not equal to the GMAC capacity region.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we show that the multiple-access nature of the channel can be utilized to improve the secrecy of the system. Allowing confidence in the secrecy of all users, the secrecy rate of a user is improved since the undecoded messages of any set of users acts as additional noise at the wire-tapper and precludes him from decoding the remaining set of users. We should note that it is possible to strengthen our definitions of secrecy as Maurer pointed out in [10] and arrive at the same results. However, we feel that the measures we have defined (as being the normalized equivocation) are conceptually easier to understand, especially as we are also interested in how we can increase the transmission rate by allowing a certain amount of information to "leak" to the wire-tapper. An interesting question would be how to control which information is leaked to the wire-tapper, and which information is kept confidential. Our results so far are based on the wire-tapper having access to a degraded version of the intended receiver's signal. The case where the eavesdropper's received signal is not necessarily degraded, where a user with a "better" channel may help provide secrecy for a user with a "worse" channel who could not otherwise communicate secretly is of current interest.

APPENDIX I

OUTER BOUNDS

We first adapt Lemma 10 in [4] to upper bound the differences between the received signal entropies at the receiver and wire-tapper:

Lemma 10 (Lemma 10 in [4]). Let $\xi = \frac{1}{n}H(\mathbf{Y})$ where \mathbf{Y}, \mathbf{Z} are as given in (4). Then,

$$H(\mathbf{Y}) - H(\mathbf{Z}) \le n\xi - n\phi(\xi) \triangleq \frac{n}{2} \log\left[2\pi e\left(1 - h + \frac{h2^{2\xi}}{2\pi e}\right)\right]$$
(36)

Corollary 10.1.

$$H(\mathbf{Y}|\mathbf{X}_{\mathcal{S}}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}}) \le \frac{n}{2} \log\left(\frac{1+P_{\mathcal{S}^c}}{1+hP_{\mathcal{S}^c}}\right)$$
(37)

Proof: The proof is easily shown using the entropy power inequality, [22]: Recall that $H(\mathbf{Z}) = H(\sqrt{h}\mathbf{Y} + \mathbf{N}^{(W-M)})$. Then, by the entropy power inequality

$$2^{\frac{2}{n}H(\mathbf{Z})} = 2^{\frac{2}{n}H(\sqrt{h}\mathbf{Y} + \mathbf{N}^{(\mathsf{W}-\mathsf{M})})} \ge 2^{\frac{2}{n}[H(\mathbf{Y}) + n\log\sqrt{h}]} + 2^{\frac{2}{n}H(\mathbf{N}^{(\mathsf{W}-\mathsf{M})})}$$
(38)

Now $H(\mathbf{Y}) = n\xi$ and $H(\mathbf{N}^{(\mathsf{W}-\mathsf{M})}) = \frac{n}{2}\log[2\pi e(1-h)]$. Hence,

$$2^{\frac{2}{n}H(\mathbf{Z})} \ge h2^{2\xi} + 2\pi e(1-h)$$
(39)

which, after taking the log, gives

$$H(\mathbf{Z}) \ge \frac{n}{2} \log \left[h 2^{2\xi} + 2\pi e(1-h) \right]$$
 (40)

$$= \frac{n}{2} \log \left[2\pi e \left(1 - h + \frac{h2^{2\xi}}{2\pi e} \right) \right]$$
(41)

subtracting from $H(\mathbf{Y}) = n\xi$ completes the proof of the lemma.

To see the corollary, write

$$H(\mathbf{Y}|\mathbf{X}_{\mathcal{S}}) \le \frac{n}{2} \log \left(2\pi e (1+P_{\mathcal{S}^c})\right)$$
(42)

Let $H(\mathbf{Y}|\mathbf{X}_{\mathcal{S}}) = n\xi$. Then, $\xi \leq \frac{1}{2}\log(2\pi e(1+P_{\mathcal{S}^c}))$, and since $\phi(\xi)$ is a non-increasing function of ξ , we get $\phi(\xi) \geq \phi\left(\frac{1}{2}\log(2\pi e(1+P_{\mathcal{S}^c}))\right)$. Since we assume $\{\mathbf{X}_i\}$ to be independent, we can use the lemma with $\mathbf{Y} \to \mathbf{Y}|\mathbf{X}_{\mathcal{S}}$ and $\mathbf{Z} \to \mathbf{Z}|\mathbf{X}_{\mathcal{S}}$,

$$H(\mathbf{Y}|\mathbf{X}_{S}) - H(\mathbf{Z}|\mathbf{X}_{S}) \le \frac{n}{2} \log \left(2\pi e(1+P_{S^{c}})\right) - \frac{n}{2} \log \left[2\pi e\left(1-h+h(1+P_{S^{c}})\right)\right]$$
(43)

$$\leq \frac{n}{2} \log \left(\frac{1 + P_{\mathcal{S}^c}}{1 + h P_{\mathcal{S}^c}} \right) \tag{44}$$

$$= n[C(P_{\mathcal{S}^c}) - C(hP_{\mathcal{S}^c})] \tag{45}$$

$$= n \left[C_{\mathcal{S}^c}^{(\mathsf{M})} - C_{\mathcal{S}^c}^{(\mathsf{W})} \right] \tag{46}$$

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A. Individual Constraints

The proof is a simple extension of the proof of Lemma 7 in [4].

$$H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z}, \mathbf{Y}) \le H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Y})$$
(47)

$$\leq n\epsilon_n$$
 (48)

where the last step follows from Fano's Inequality with $\epsilon_n = h(P_e) + P_e \log \left(\prod_{i \in S} M_i - 1\right)$. Using the definition of $\Delta_S^{(I)}$, and $R_S = \frac{1}{n} \sum_{j \in S} H(W_j) = \frac{1}{n} H(\mathbf{W}_S)$ we write

$$nR_{\mathcal{S}}\Delta_{\mathcal{S}}^{(I)} = H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^c}, \mathbf{Z})$$
(49)

From (48), we have $n\epsilon_n - H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^c},\mathbf{Z},\mathbf{Y}) \geq 0$. Thus,

$$nR_{\mathcal{S}}\Delta_{\mathcal{S}}^{(I)} \le H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z}) + n\epsilon_{n} - H(\mathbf{W}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z}, \mathbf{Y})$$
(50)

$$= I(\mathbf{W}_{\mathcal{S}}; \mathbf{Y} | \mathbf{X}_{\mathcal{S}^c}, \mathbf{Z}) + n\epsilon_n$$
(51)

$$\leq I(\mathbf{X}_{\mathcal{S}}; \mathbf{Y} | \mathbf{X}_{\mathcal{S}^c}, \mathbf{Z}) + n\epsilon_n \tag{52}$$

$$= H(\mathbf{X}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Y}, \mathbf{Z}) + n\epsilon_{n}$$
(53)

$$= H(\mathbf{X}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}},\mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}},\mathbf{Y}) + n\epsilon_{n}$$
(54)

Repeatedly using H(A, B) = H(A) + H(B|A), we can write

$$nR_{\mathcal{S}}\Delta_{\mathcal{S}}^{(I)} \leq \left[H(\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z} | \mathbf{X}_{\mathcal{S}}) - H(\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Y} | \mathbf{X}_{\mathcal{S}})\right] - \left[H(\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Y})\right]$$
(55)

$$\leq [H(\mathbf{Z}|\mathbf{X}_{\mathcal{K}}) - H(\mathbf{Y}|\mathbf{X}_{\mathcal{K}})] - [H(\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}^{c}}, \mathbf{Y})]$$
(56)

$$= \frac{n}{2} \log \left(2\pi e\right) - \frac{n}{2} \log \left(2\pi e\right) - \left[H(\mathbf{X}_{\mathcal{S}^c}, \mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}^c}, \mathbf{Y})\right]$$
(57)

$$= H(\mathbf{Y}|\mathbf{X}_{\mathcal{S}^c}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}^c})$$
(58)

Using Corollary 10.1, we arrive at

$$nR_{\mathcal{S}}\Delta_{\mathcal{S}}^{(I)} \le nC_{\mathcal{S}}^{(\mathsf{M})} - nC_{\mathcal{S}}^{(\mathsf{W})}$$
(59)

Since we want $\Delta_{\mathcal{S}}^{(I)} \geq \delta$, this gives us

$$R_{\mathcal{S}} \le \frac{1}{\delta} \left[C_{\mathcal{S}}^{(\mathsf{M})} - C_{\mathcal{S}}^{(\mathsf{W})} \right] \tag{60}$$

B. Collective Constraints

We show that any achievable rate vector, \mathbf{R} , needs to satisfy Theorem 3. The first term in the minimum of (20) is due to the converse of the GMAC coding theorem. To see the second constraint, we start with a few lemmas:

Lemma 11. Let $\mathbf{X}_{S} = {\{\mathbf{X}_{k}\}_{k \in S}}$ where $S \subseteq \mathcal{K}$. Then,

$$R_{\mathcal{S}} \leq \frac{1}{n\delta} I(\mathbf{X}_{\mathcal{S}}; \mathbf{Y} | \mathbf{Z}) + \nu_n \quad \forall \mathcal{S} \subseteq \mathcal{K}$$
(61)

where $\nu_n \to 0$ as $\epsilon \to 0$.

Proof: Let $S \subseteq K$ and consider the two inequalities:

$$\delta \leq \Delta_{\mathcal{S}}^{(C)} = \frac{H(\mathbf{W}_{\mathcal{S}}|\mathbf{Z})}{\log\left(\prod_{j\in\mathcal{S}} M_{j}\right)} \leq \frac{H(\mathbf{W}_{\mathcal{S}}|\mathbf{Z})}{n\left(R_{\mathcal{S}} - |\mathcal{S}|\epsilon\right)}$$
(62)

$$H(\mathbf{W}_{\mathcal{S}}|\mathbf{Z},\mathbf{Y}) \le H(\mathbf{W}_{\mathcal{S}}|\mathbf{Y}) \le H(\mathbf{W}_{\mathcal{K}}|\mathbf{Y}) \le \eta_n$$
(63)

where (63) follows using Fano's Inequality with $\eta_n \to 0$ as $\epsilon \to 0$ and $n \to \infty$. Using (62) and (63), we can write

$$\delta \leq \frac{H(\mathbf{W}_{\mathcal{S}}|\mathbf{Z}) + \eta_n - H(\mathbf{W}_{\mathcal{S}}|\mathbf{Z}, \mathbf{Y})}{n\left(R_{\mathcal{S}} - |\mathcal{S}|\epsilon\right)}$$
(64)

$$\leq \frac{I(\mathbf{X}_{\mathcal{S}}; \mathbf{Y} | \mathbf{Z}) + \eta_n}{n \left(R_{\mathcal{S}} - |\mathcal{S}| \epsilon \right)}$$
(65)

with the last step using $W_{\mathcal{S}} \to \mathbf{X}_{\mathcal{S}} \to \mathbf{Y} \to \mathbf{Z}$. Rearranging and defining $\nu_n \triangleq \frac{\eta_n}{n\delta} + |\mathcal{S}|\epsilon$ completes the proof.

Lemma 12. For the GMAC-WT,

$$I(\mathbf{X}_{\mathcal{S}}; \mathbf{Y} | \mathbf{Z}) \le nC_{\mathcal{S}}^{(\mathsf{M})} - nC\left(\frac{\frac{h}{2\pi e}\sum_{j\in\mathcal{S}} 2^{\frac{2}{n}H(\mathbf{X}_j)}}{hP_{\mathcal{S}^c} + 1}\right)$$
(66)

Corollary 12.1. For the GMAC-WT,

$$I(\mathbf{X}_{\mathcal{K}}; \mathbf{Y} | \mathbf{Z}) \le n \left(C_{sum}^{(\mathsf{M})} - C_{sum}^{(\mathsf{W})} \right)$$
(67)

Proof: Start by writing

$$I(\mathbf{X}_{\mathcal{S}}, \mathbf{Y} | \mathbf{Z}) = H(\mathbf{X}_{\mathcal{S}} | \mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}} | \mathbf{Y}, \mathbf{Z})$$
(68)

$$= H(\mathbf{X}_{\mathcal{S}}|\mathbf{Z}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{Y})$$
(69)

$$= [H(\mathbf{X}_{\mathcal{S}}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{Y})] - [H(\mathbf{X}_{\mathcal{S}}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{Z})]$$
(70)

$$\leq \left[H(\mathbf{X}_{\mathcal{S}}|\mathbf{X}_{\mathcal{S}^{c}}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{Y},\mathbf{X}_{\mathcal{S}^{c}})\right] - \left[H(\mathbf{X}_{\mathcal{S}}) - H(\mathbf{X}_{\mathcal{S}}|\mathbf{Z})\right]$$
(71)

$$= I(\mathbf{X}_{\mathcal{S}}; \mathbf{Y} | \mathbf{X}_{\mathcal{S}^c}) - I(\mathbf{X}_{\mathcal{S}}; \mathbf{Z})$$
(72)

$$= H(\mathbf{Y}|\mathbf{X}_{\mathcal{S}^c}) - H(\mathbf{Y}|\mathbf{X}_{\mathcal{K}}) - [H(\mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})]$$
(73)

$$= \sum_{i=1}^{n} H(Y_i|Y^{i-1}, \mathbf{X}_{\mathcal{S}^c}) - \sum_{i=1}^{n} H(Y_i|Y^{i-1}, \mathbf{X}_{\mathcal{K}}) - [H(\mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})]$$
(74)

$$\leq \sum_{i=1}^{n} H(Y_i | \mathbf{X}_{\mathcal{S}^c, i}) - \sum_{i=1}^{n} H(Y_i | \mathbf{X}_{\mathcal{K}, i}) - [H(\mathbf{Z}) - H(\mathbf{Z} | \mathbf{X}_{\mathcal{S}})]$$
(75)

$$\leq \sum_{i=1}^{n} \frac{1}{2} \log \left[2\pi e \left(1 + P_{\mathcal{S}} \right) \right] - \sum_{i=1}^{n} \frac{1}{2} \log \left(2\pi e \right) - \left[H(\mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}}) \right]$$
(76)

$$= nC(P_{\mathcal{S}}) - [H(\mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})]$$
(77)

where (69) follows from $X_S \to Y \to Z$ and (75) follows using the memoryless property of M. For the term in brackets, start by using the entropy power inequality:

$$2^{\frac{2}{n}H(\mathbf{Z})} \ge 2^{\frac{2}{n}H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})} + \sum_{j\in\mathcal{S}} 2^{\frac{2}{n}H(\sqrt{h}\mathbf{X}_j)}$$
(78)

$$2^{\frac{2}{n}H(\mathbf{Z}) - \frac{2}{n}H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})} \ge 1 + 2^{-\frac{2}{n}H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})} \sum_{j \in \mathcal{S}} 2^{\frac{2}{n}\left[H(\mathbf{X}_j) + n\log\sqrt{h}\right]}$$
(79)

$$= 1 + h2^{-\frac{2}{n}H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}})} \sum_{j \in \mathcal{S}} 2^{\frac{2}{n}H(\mathbf{X}_j)}$$
(80)

and

$$2^{\frac{2}{n}H(\mathbf{Z}|\mathbf{X}_{S})} = 2^{\frac{2}{n}\sum_{i=1}^{n}H(Z_{i}|Z^{i-1},\mathbf{X}_{S})}$$
(81)

$$\leq 2^{\frac{2}{n}\sum_{i=1}^{n}H(Z_{i}|\mathbf{X}_{\mathcal{S},i})}$$
(82)

$$\leq 2^{\frac{2}{n}\sum_{i=1}^{n}\frac{1}{2}\log(2\pi e(hP_{\mathcal{S}^{c}}+1))}$$
(83)

$$=2\pi e(hP_{\mathcal{S}^c}+1) \tag{84}$$

Using this in (80), and taking the log we get,

$$H(\mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{\mathcal{S}}) \ge \frac{n}{2} \log \left(1 + \frac{\frac{h}{2\pi e} \sum_{j \in \mathcal{S}} 2^{\frac{2}{n}H(\mathbf{X}_j)}}{hP_{\mathcal{S}^c} + 1} \right)$$
(85)

which, with (77) completes the proof. To see the corollary,

$$I(\mathbf{X}_{\mathcal{K}}; \mathbf{Y} | \mathbf{Z}) = H(\mathbf{X}_{\mathcal{K}} | \mathbf{Z}) - H(\mathbf{X}_{\mathcal{K}} | \mathbf{Y}, \mathbf{Z})$$
(86)

$$= H(\mathbf{X}_{\mathcal{K}}|\mathbf{Z}) - H(\mathbf{X}_{\mathcal{K}}|\mathbf{Y})$$
(87)

$$= [H(\mathbf{Z}|\mathbf{X}_{\mathcal{K}}) + H(\mathbf{X}_{\mathcal{K}}) - H(\mathbf{Z})] - [H(\mathbf{Y}|\mathbf{X}_{\mathcal{K}}) + H(\mathbf{X}_{\mathcal{K}}) - H(\mathbf{Y})]$$
(88)

$$= [H(\mathbf{Z}|\mathbf{X}_{\mathcal{K}}) - H(\mathbf{Y}|\mathbf{X}_{\mathcal{K}})] - [H(\mathbf{Z}) - H(\mathbf{Y})]$$
⁽⁸⁹⁾

$$=\sum_{i=1}^{n} \left[H(Z_i|\mathbf{X}_{\mathcal{K},i}) - H(Y_i|\mathbf{X}_{\mathcal{K},i})\right] - \left[H(\mathbf{Z}) - H(\mathbf{Y})\right]$$
(90)

$$= \left[\frac{n}{2}\log\left(2\pi e\right) - \frac{n}{2}\log\left(2\pi e\right)\right] - \left[H(\mathbf{Z}) - H(\mathbf{Y})\right]$$
(91)

$$=H(\mathbf{Y})-H(\mathbf{Z})\tag{92}$$

$$\leq C_{sum}^{(M)} - C_{sum}^{(W)} \tag{93}$$

where (87) is due to $X_{\mathcal{K}} \to Y \to Z$ and (90) to the memorylessness of the channels. (93) follows from Corollary 10.1.

This and Lemma 11, complete the proof of Theorem 3.

Corollary 3.1 follows from Corollary 12.1 and Lemma 11.

Corollary 3.2 follows simply with $H(\mathbf{X}_j) = \frac{n}{2} \log 2\pi e P_j$.

APPENDIX II

ACHIEVABLE RATES

A. Individual Constraints

Let $\mathbf{R} = (R_1, ..., R_K)$ satisfy (28).

For each user $k \in \mathcal{K}$, consider the scheme:

- 1) Let $M_k = 2^{n(R_k \epsilon')}$ where $0 \le \epsilon' < \epsilon$. Let $M_k = M_{ks}M_{k0}$ where $M_{ks} = M_k^{\mu_k}$, $M_{k0} = M_k^{1-\mu_k}$, and $1 \ge \mu_k \ge \delta$ will be chosen later. Then, $R_k = R_{ks} + R_{k0} + \epsilon'$ where $R_{ks} = \frac{1}{n} \log M_{ks}$ and $R_{k0} = \frac{1}{n} \log M_{k0}$. We can choose ϵ' and n to ensure that M_{ks}, M_{k0} are integers.
- 2) Generate 3 codebooks X_{ks}, X_{k0} and X_{kx}. X_{ks} consists of M_{ks} codewords, each component of which is drawn ~ N (0, λ_{ks}P_k − ε). Codebook X_{k0} has M_{k0} codewords with each component randomly drawn ~ N (0, λ_{k0}P_k − ε) and X_{kx} has M_{kx} codewords with each component

randomly drawn ~ $\mathcal{N}(0, \lambda_{kx}P_k - \varepsilon)$ where ε is an arbitrarily small number to ensure that the power constraints on the codewords are satisfied with high probability and $\lambda_{ks} + \lambda_{k0} + \lambda_{kx} = 1$. Define $R_{kx} = \frac{1}{n} \log M_{kx}$ and $M_{kt} = M_k M_{kx}$.

- 3) Each message W_k ∈ {1,..., M_k} is mapped into a message vector W_k = (W_{ks}, W_{k0}) where W_{ks} ∈ {1,..., M_{ks}} and W_{k0} ∈ {1,..., M_{k0}}. Since W_k is uniformly chosen, W_{ks}, W_{k0} are also uniformly distributed.
- 4) To transmit message W_k ∈ {1,..., M_k}, user k finds the 2 codewords corresponding to components of W_k and also uniformly chooses a codeword from X_{kx}. He then adds all these codewords and transmits the resulting codeword, X_k, so that we are actually transmitting one of M_{kt} codewords. Let R_{kt} = ¹/_n log M_{kt} + ε' = R_{ks} + R_{k0} + R_{kx} + ε'.

Specifically, the rates are chosen to satisfy $\forall S \subseteq \mathcal{K}$:

$$\sum_{k \in \mathcal{S}} R_{ks} = \sum_{k \in \mathcal{S}} \mu_k R_k \le C_{\mathcal{S}}^{(\mathsf{M})} - \sum_{k \in \mathcal{S}} C_k^{(\mathsf{W})}$$
(94)

$$R_{k0} + R_{kx} = (1 - \mu_k)R_k + R_{kx} = C_k^{(W)}, \quad \forall k \in \mathcal{S}$$
(95)

$$\sum_{k \in \mathcal{S}} R_{kt} = \sum_{k \in \mathcal{S}} [R_k + R_{kx}] \le C_{\mathcal{S}}^{(\mathsf{M})}$$
(96)

Consider the subcode $\{\mathfrak{X}_{ks}\}_{k=1}^{K}$. From this point of view, the coding scheme described is equivalent to each user $k \in \mathcal{K}$ selecting one of M_{ks} messages, and sending a uniformly chosen codeword from among $M_{k0}M_{kx}$ codewords for each. Let $\Delta_{k}^{(i)} = \frac{H(W_{ks}|\mathbf{X}_{k^{c}},\mathbf{Z})}{H(W_{ks})}$ and write the following:

$$H(W_{ks}|\mathbf{X}_{k^c}, \mathbf{Z}) = H(W_{ks}, \mathbf{X}_{k^c}, \mathbf{Z}) - H(\mathbf{X}_{k^c}, \mathbf{Z})$$
(97)

$$= H(W_{ks}, \mathbf{X}_k, \mathbf{X}_{k^c}, \mathbf{Z}) - H(\mathbf{X}_k | W_{ks}, \mathbf{X}_{k^c}, \mathbf{Z}) - H(\mathbf{X}_{k^c}, \mathbf{Z})$$
(98)

$$= H(\mathbf{Z}|W_{ks}, \mathbf{X}_{k}, \mathbf{X}_{k^{c}}) + H(\mathbf{X}_{k}, \mathbf{X}_{k^{c}}|W_{ks}) + H(W_{ks})$$
$$- H(\mathbf{X}_{k}|W_{ks}, \mathbf{X}_{k^{c}}, \mathbf{Z}) - H(\mathbf{X}_{k^{c}}, \mathbf{Z})$$
$$= H(\mathbf{Z}|\mathbf{X}_{k}, \mathbf{X}_{k^{c}}) + H(\mathbf{X}_{k}, \mathbf{X}_{k^{c}}|W_{ks}) + H(W_{ks})$$
(99)

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$$-H(\mathbf{X}_{k}|W_{ks},\mathbf{X}_{k^{c}},\mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{k^{c}}) - H(\mathbf{X}_{k^{c}})$$
(100)

$$= H(\mathbf{Z}|\mathbf{X}_{k}, \mathbf{X}_{k^{c}}) + H(\mathbf{X}_{k^{c}}|W_{ks}) + H(\mathbf{X}_{k}|\mathbf{X}_{k^{c}}, W_{ks}) + H(W_{k})$$
$$- H(\mathbf{X}_{k}|W_{ks}, \mathbf{X}_{k^{c}}, \mathbf{Z}) - H(\mathbf{Z}|\mathbf{X}_{k^{c}}) - H(\mathbf{X}_{k^{c}})$$
(101)

$$= H(W_{ks}) - I(\mathbf{X}_k; \mathbf{Z} | \mathbf{X}_{k^c}) + I(\mathbf{X}_k; \mathbf{Z} | W_{ks}, \mathbf{X}_{k^c})$$
(102)

Using this in the definition of $\Delta_k^{(i)},$ we can write

$$\Delta_k^{(i)} = 1 - \frac{I(\mathbf{X}_k; \mathbf{Z} | \mathbf{X}_{k^c}) - I(\mathbf{X}_k; \mathbf{Z} | W_{ks}, \mathbf{X}_{k^c})}{H(W_{ks})}$$
(103)

By the GMAC coding theorem, we have $I(\mathbf{X}_j; \mathbf{Z} | \mathbf{X}_{j^c}) \leq nC_j^{(W)}$. We can also write

$$I(\mathbf{X}_k; \mathbf{Z} | W_{ks}, \mathbf{X}_{k^c}) = H(\mathbf{X}_k | W_{ks}, \mathbf{X}_{k^c}) - H(\mathbf{X}_k | W_{ks}, \mathbf{X}_{k^c}, \mathbf{Z})$$
(104)

Our coding scheme implies that

$$H(\mathbf{X}_k|W_{ks}, \mathbf{X}_{k^c}) = H(\mathbf{X}_k|W_{ks}) = nC_k^{(\mathsf{W})}$$
(105)

Also,

$$H(\mathbf{X}_k|W_{ks}, \mathbf{X}_{k^c}, \mathbf{Z}) \le n\delta_n \tag{106}$$

where $\delta_n \to 0$ due to Fano's Inequality. This stems from the fact that given W_{ks} , the subcode for user k is, with high probability, a "good" code for the wiretapper. Combining these in (103), we can write

$$\Delta_k^{(i)} \ge 1 - \frac{nC_k^{(M)} - nC_k^{(W)} + n\delta_n}{H(W_{ks})}$$
(107)

$$=1-\frac{n\delta_n}{nR_{ks}}\tag{108}$$

$$= 1 - \epsilon \tag{109}$$

where $\epsilon = \frac{\delta_n}{R_{ks}} \to 0$ as $n \to \infty$.

Then, we can write

$$\Delta_{k}^{(I)} = \frac{H(W_{k} | \mathbf{X}_{k^{c}}, \mathbf{Z})}{H(W_{k})} \ge \frac{H(W_{ks} | \mathbf{X}_{k^{c}}, \mathbf{Z})}{H(W_{k})} \ge \frac{(1 - \epsilon)H(W_{ks})}{H(W_{k})} = \frac{(1 - \epsilon)R_{ks}}{R_{k}} \ge \frac{(1 - \epsilon)\mu_{k}R_{k}}{R_{k}} \ge \delta$$
(110)

Since (110) holds for all k = 1, ..., K, from (12) we have $\Delta_{\mathcal{S}}^{(I)} \ge \delta, \forall \mathcal{S} \subseteq \mathcal{K}$.

B. Collective Constraints

Let $\mathbf{R} = (R_1, \dots, R_K)$ satisfy (22) and assume the coding scheme is the same as described in the individual constraints case.

We will choose the rates such that for all $S \subseteq \mathcal{K}$,

$$\sum_{k \in \mathcal{S}} R_{ks} = \sum_{k \in \mathcal{S}} \mu_k R_k \le C_{\mathcal{S}}^{(\mathsf{M})} - \tilde{C}_{\mathcal{S}}^{(\mathsf{W})}$$
(111)

$$\sum_{k=1}^{K} [R_{k0} + R_{kx}] = \sum_{k=1}^{K} [(1 - \mu_k)R_k + R_{kx}] = C_{sum}^{(W)}$$
(112)

$$\sum_{k \in \mathcal{S}} R_{kt} = \sum_{k \in \mathcal{S}} [R_k + R_{kx}] \le C_{\mathcal{S}}^{(\mathsf{M})}$$
(113)

From (113) and the GMAC coding theorem, with high probability the receiver can decode the codewords with low probability of error. To show $\Delta_{\mathcal{S}} \geq \delta$, $\forall \mathcal{S} \subseteq \mathcal{K}$, we concern ourselves only with MAC sub-code $\{\mathfrak{X}_{ks}\}_{k=1}^{K}$. From this point of view, the coding scheme described is equivalent to each user $k \in \mathcal{K}$ selecting one of M_{ks} messages, and sending a uniformly chosen codeword from among $M_{k0}M_{kx}$ codewords for each. Let $\mathbf{W}_{\mathcal{S}}^{(s)} = \{W_{ks}\}_{k\in\mathcal{S}}$ and $\Delta_{\mathcal{S}}^{(s)} = \frac{H(\mathbf{W}_{\mathcal{S}}^{(s)}|\mathbf{Z})}{H(\mathbf{W}_{\mathcal{S}}^{(s)})}$ and define $\mathbf{X}_{\Sigma} = \sum_{k=1}^{K} \mathbf{X}_{k}$. For \mathcal{K} write

$$\Delta_{\mathcal{K}}^{(s)} = \frac{H(\mathbf{W}_{\mathcal{K}}^{(s)}|\mathbf{Z})}{H(\mathbf{W}_{\mathcal{K}}^{(s)})}$$
(114)

$$=\frac{H(\mathbf{W}_{\mathcal{K}}^{(s)}, \mathbf{Z}) - H(\mathbf{Z})}{H(\mathbf{W}_{\mathcal{K}}^{(s)})}$$
(115)

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$$=\frac{H(\mathbf{W}_{\mathcal{K}}^{(s)}, \mathbf{X}_{\Sigma}, \mathbf{Z}) - H(\mathbf{X}_{\Sigma} | \mathbf{W}_{\mathcal{K}}^{(s)}, \mathbf{Z}) - H(\mathbf{Z})}{H(\mathbf{W}_{\mathcal{K}}^{(s)})}$$
(116)

$$=\frac{H(\mathbf{W}_{\mathcal{K}}^{(s)})+H(\mathbf{Z}|\mathbf{W}_{\mathcal{K}}^{(s)},\mathbf{X}_{\Sigma})-H(\mathbf{Z})}{H(\mathbf{W}_{\mathcal{K}}^{(s)})}+\frac{H(\mathbf{X}_{\Sigma}|\mathbf{W}_{\mathcal{K}}^{(s)})-H(\mathbf{X}_{\Sigma}|\mathbf{W}_{\mathcal{K}}^{(s)},\mathbf{Z})}{H(\mathbf{W}_{\mathcal{K}}^{(s)})}$$
(117)

$$= 1 - \frac{I(\mathbf{X}_{\Sigma}; \mathbf{Z}) - I(\mathbf{X}_{\Sigma}; \mathbf{Z} | \mathbf{W}_{\mathcal{K}}^{(s)})}{n\left(\sum_{k=1}^{K} R_{ks}\right)}$$
(118)

where we used $\mathbf{W}_{\mathcal{K}}^{(s)} \to \mathbf{X}_{\Sigma} \to \mathbf{Z} \Rightarrow H(\mathbf{Z}|\mathbf{W}_{\mathcal{K}}^{(s)}, \mathbf{X}_{\Sigma}) = H(\mathbf{Z}|\mathbf{X}_{\Sigma})$ to get (118). We will consider the two terms individually. First, we have the trivial bound due to channel capacity:

$$I(\mathbf{X}_{\Sigma}; \mathbf{Z}) \le nC_{sum}^{(\mathsf{W})} \tag{119}$$

 $I(\mathbf{X}_{\Sigma}; \mathbf{Z} | \mathbf{W}_{\mathcal{K}}^{(s)}) = H(\mathbf{X}_{\Sigma} | \mathbf{W}_{\mathcal{K}}^{(s)}) - H(\mathbf{X}_{\Sigma} | \mathbf{W}_{\mathcal{K}}^{(s)}, \mathbf{Z}).$ Since user k sends one of $M_{k0}M_{kx}$ codewords for each message,

$$H(\mathbf{X}_{\Sigma}|\mathbf{W}_{\mathcal{K}}^{(s)}) = \log\left(\prod_{k=1}^{K} M_{k0} M_{kx}\right)$$
(120)

$$= n \sum_{k=1}^{K} [(1 - \mu_k)R_k + R_{kx}]$$
(121)

We can also write

$$H(\mathbf{X}_{\Sigma}|\mathbf{W}_{\mathcal{K}}^{(s)},\mathbf{Z}) \le n\eta_{n}^{\prime}$$
(122)

where $\eta'_n \to 0$ as $n \to \infty$ since, with high probability, the eavesdropper can decode \mathbf{X}_{Σ} given $\mathbf{W}_{\mathcal{K}}^{(s)}$ due to (112). Using (111), (112), (119), (121) and (122) in (118), we get

$$\Delta_{\mathcal{K}}^{(s)} \ge 1 - \frac{C_{sum}^{(\mathsf{W})} - \sum_{k=1}^{K} \left[(1 - \mu_k) R_k + R_{kx} \right] + \eta'_n}{C_{sum}^{(\mathsf{M})} - C_{sum}^{(\mathsf{W})}}$$
(123)

$$= 1 - \frac{\eta'_n}{C_{sum}^{(\mathsf{M})} - C_{sum}^{(\mathsf{W})}} \to 1 \text{ as } \eta'_n \to 0$$
(124)

Then,

$$H(\mathbf{W}_{\mathcal{K}}^{(s)}|\mathbf{Z}) = H(\mathbf{W}_{\mathcal{K}}^{(s)})$$
(125)

$$H(\mathbf{W}_{\mathcal{S}}^{(s)}|\mathbf{Z}) + H(\mathbf{W}_{\mathcal{S}^{c}}^{(s)}|\mathbf{Z}) \ge H(\mathbf{W}_{\mathcal{S}}^{(s)}) + H(\mathbf{W}_{\mathcal{S}^{c}}^{(s)})$$
(126)

As conditioning reduces entropy, we have $H(\mathbf{W}_{\mathcal{S}}^{(s)}|\mathbf{Z}) \leq H(\mathbf{W}_{\mathcal{S}}^{(s)})$ and $H(\mathbf{W}_{\mathcal{S}^{c}}^{(s)}|\mathbf{Z}) \leq H(\mathbf{W}_{\mathcal{S}^{c}}^{(s)})$.

Then, from the above equation we conclude that we must have $H(\mathbf{W}_{S}^{(s)}) = H(\mathbf{W}_{S}^{(s)}|\mathbf{Z}), \ \forall S \subset \mathcal{K}.$ This makes $\Delta_{S}^{(s)} = 1 \ \forall S \subset \mathcal{K}.$ The proof is completed by noting that

$$\Delta_{\mathcal{S}} \ge \frac{H(\mathbf{W}_{\mathcal{S}}^{(s)}|\mathbf{Z})}{H(\mathbf{W}_{\mathcal{S}})} = \frac{H(\mathbf{W}_{\mathcal{S}}^{(s)})}{H(\mathbf{W}_{\mathcal{S}})} = \frac{\sum_{k \in \mathcal{S}} \mu_k R_k}{\sum_{k \in \mathcal{S}} R_k} \ge \delta$$
(127)

We can think of $\{W_{ks}\}$ as the set of "protected" messages and $\{W_{k0}\}$ as the set of "unprotected" messages. Note that this is more of a conceptual difference, since the same argument can be made about any such subset of the transmitted messages. In other words, we can be assured of the perfect secrecy of a subset of messages of size $2^{nR_{ks}}$ for each user, but not necessarily which subset. The corollary was shown in the steps to get (124), and also follows easily as (22) simplifies to $\mathbf{R}_{S} \leq C_{S}^{(M)} - \tilde{C}_{S}^{(W)}$ if $\delta = 1$.

APPENDIX III

SUM CAPACITY MAXIMIZATION

A. Individual Constraints

Proof: [Proof of Theorem 5] The optimization problem given in (30) can be written as:

$$\max_{\substack{\{P_i\}\\ \text{s. t.}}} \hat{R}(\mathbf{P})$$

s. t. $0 \le P_i \le P_{i,max}$

where

$$\hat{R}(\mathbf{P}) \triangleq \frac{1}{2} \log \left(1 + \sum_{j=1}^{K} P_j \right) - \sum_{j=1}^{K} \frac{1}{2} \log \left(1 + h P_j \right)$$
(128)

We can write the Lagrangian of the corresponding *minimization* problem as

$$\mathcal{L}(\mathbf{P}) = -\frac{1}{2}\log\left(1 + \sum_{j=1}^{K} P_j\right) + \sum_{j=1}^{K} \frac{1}{2}\log\left(1 + hP_j\right) - \sum_{j=1}^{K} \mu_{1j}P_j + \sum_{j=1}^{K} \mu_{2j}(P_j - P_{j,max})$$
(129)

If we take the derivative with respect to P_k , we get

$$\frac{\partial \mathcal{L}}{\partial P_k} = -\frac{\ln 2}{2} \left(\frac{1}{1 + \sum_{j=1}^K P_j} - \frac{h}{1 + hP_k} \right) - \mu_{1k} + \mu_{2k} = 0$$
(130)

Also note the second derivatives:

$$\frac{\partial^2 \mathcal{L}}{\partial P_k \partial P_l} = \frac{\ln 2}{2} \frac{1}{\left(1 + \sum_{j=1}^K P_j\right)^2}, \quad l \neq k$$
(131)

$$\frac{\partial^2 \mathcal{L}}{\partial P_k^2} = \frac{\ln 2}{2} \left(\frac{1}{\left(1 + \sum_{j=1}^K P_j\right)^2} - \frac{h^2}{\left(1 + hP_k\right)^2} \right)$$
(132)

Assume, for some k, that $0 < P_k^* < P_{k,max}$. Then, $\mu_{1k} = \mu_{2k} = 0$, and

$$\frac{\partial \mathcal{L}}{\partial P_k} = 0 \Rightarrow \sum_{j \neq k} P_j = \frac{1}{h} - 1$$
(133)

However, looking at the Jacobian in this case gives $\frac{\partial^2 \mathcal{L}}{\partial P_k \partial P_l} > 0$ is constant for all k, l, and $\frac{\partial^2 \mathcal{L}}{\partial P_k^2} = 0$ for all k. Then, this point cannot be an optimum point as the Jacobian is neither positive nor negative semi-definite. As a result, we see that the optimum power allocation will lie on the boundaries, i.e., P_k^* will either be 0 or $P_{k,max}$ for any $k \in \mathcal{K}$.

To see the second part, assume \mathcal{T} is the set of transmitting users, i.e., $k \in \mathcal{T} \Rightarrow P_k = P_{k,max}$ and $k \notin \mathcal{T} \Rightarrow P_k = 0$. For all $k \in \mathcal{T}$, we then have $\mu_{1k} = 0 \Rightarrow \frac{\partial \hat{R}(\mathbf{P})}{\partial P_k} > 0$, and for all $k \notin \mathcal{T}$, $\mu_{2k} = 0 \Rightarrow \frac{\partial \hat{R}(\mathbf{P})}{\partial P_k} < 0$. Using this in (130) completes the proof.

This, in general, does not lead to any closed form solutions. However, the following special cases are notable:

- If $\frac{1}{h} 1 \le P_{1,max} \le P_{2,max} \le \ldots \le P_{K,max}$, then only user K transmits.
- If $\sum_{j=1}^{K} P_{j,max} \leq \frac{1}{h} 1$, then all users transmit with maximum power.
- If $P_{1,max} \leq P_{2,max} \leq \ldots \leq \frac{1}{h} 1 \leq P_{j,max} \leq \ldots \leq P_{K,max}$, then either
 - a subset of users from the set $\{1,\ldots,j-1\}$ will transmit with full power, or
 - user K transmits with maximum power.

B. TDMA

Maximizing (34) over the time-sharing parameters $\{\alpha_k\}$, is a convex optimization problem over α_k . Taking the derivative of the Lagrangian with respect to α_k and equating it to zero gives

$$\alpha_k^* = \frac{P_{k,max}}{\sum_{j=1}^K P_{j,max}} \tag{134}$$

Using this in (34) completes the proof.



Fig. 1. Equivalent GMAC-WT System Model for the degraded case.



Fig. 2. Regions for $P_1 = 10, P_2 = 5, \delta = 0.01$ and h = 0.9



Fig. 3. Regions for $P_1 = 10$, $P_2 = 5$, $\delta = 0.01$ and h = 0.5



Fig. 4. Regions for $P_1 = 10, P_2 = 5, \delta = 0.01$ and h = 0.1



Fig. 5. Regions for $P_1 = 10$, $P_2 = 5$, $\delta = 0.5$ and h = 0.9



Fig. 6. Regions for $P_1 = 10, P_2 = 5, \delta = 0.5$ and h = 0.5



Fig. 7. Regions for $P_1 = 10$, $P_2 = 5$, $\delta = 0.5$ and h = 0.1



Fig. 8. Regions for $P_1 = 10, P_2 = 5, \delta = 1$ and h = 0.9



Fig. 9. Regions for $P_1 = 10, P_2 = 5, \delta = 1$ and h = 0.5



Fig. 10. Regions for $P_1 = 10, P_2 = 5, \delta = 1$ and h = 0.1



Fig. 11. The two-user rate region as a function of δ with collective constraints. h = 0.2.



Fig. 12. The two-user rate region as a function of δ with individual constraints. h = 0.2.



Fig. 13. The two-user rate region as a function of δ with collective constraints. h = 0.5.



Fig. 14. The two-user rate region as a function of δ with individual constraints. h = 0.5.



Fig. 15. The two-user rate region as a function of h with collective constraints. $\delta = 1$.



Fig. 16. The two-user rate region as a function of h with individual constraints. $\delta = 1$.



Fig. 17. The two-user rate region as a function of h with collective constraints. $\delta = 0.5$.



Fig. 18. The two-user rate region as a function of h with individual constraints. $\delta = 0.5$.

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